Degree and class of caustics by reflection for a generic source

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DEGREE AND CLASS OF CAUSTICS BY REFLECTION FOR A GENERIC SOURCE

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Abstract. Given any irreducible algebraic (mirror) curve $\mathcal{C} \subseteq \mathbb{P}^2 := \mathbb{P}^2(\mathbb{C})$ and any (light position) $S \in \mathbb{P}^2$, the caustic by reflection $\Sigma_S(\mathcal{C})$ of $\mathcal{C}$ from S is the Zariski closure of the envelope of the reflected lines got from the lines coming from S after reflection on $\mathcal{C}$. In [7, 8], we established formulas for the degree and class (with multiplicity) of $\Sigma_S(\mathcal{C})$ for any $\mathcal{C}$ and any S. In this paper, we prove the birationality of the caustic map for a generic S in $\mathbb{P}^2$. Moreover, we give simple formulas for the degree and class (without multiplicity) of $\Sigma_S(\mathcal{C})$ for any $\mathcal{C}$ and for a generic S in $\mathbb{P}^2$.

1. Introduction

We are interested in the study of caustics by reflection in the projective complex plane $\mathbb{P}^2$. Given an irreducible algebraic curve $\mathcal{C} = V(F) \subset \mathbb{P}^2$ of degree $d \geq 2$ and given $S = [x_0 : y_0 : z_0] \in \mathbb{P}^2$, the **caustic by reflection** $\Sigma_S(\mathcal{C})$ of $\mathcal{C}$ from S is the Zariski closure of the envelope of the reflected lines on $\mathcal{C}$ of the lines coming from S.

For $m \in \mathcal{C}$, the reflected line $\mathcal{R}_{m,S,\mathcal{C}}$ is defined as the orthogonal symmetric of the (incident) line $(m\,S)$ with respect to the tangent line to $\mathcal{C}$ at m. In [7, 8], we detail the construction of the reflected lines and we define two rational maps $\rho_{F,S}$ and $\Phi_{F,S}$ from $\mathbb{P}^2$ into itself satisfying the following property: For a generic m in $\mathcal{C}$, $\rho_{F,S}(m)$ corresponds to an equation of the reflected line $\mathcal{R}_{m,S,\mathcal{C}}$ and this line is tangent to $\Phi_{F,S}(\mathcal{C})$ at $\Phi_{F,S}(m)$. Hence the caustic $\Sigma_S(\mathcal{C})$ is the Zariski closure of $\Phi_{F,S}(\mathcal{C})$ and $\Phi_{F,S}$ is called the **caustic map** of $\mathcal{C}$ from S. Observe that the Zariski closure of $\rho_{F,S}(\mathcal{C})$ is then the dual curve of the caustic $\Sigma_S(\mathcal{C})$. In [7, 8], we used this approach to establish precise formulas for the degree anf class (both with multiplicity) of $\Sigma_S(\mathcal{C})$ for any $\mathcal{C}$ and any S. The degree with multiplicity of $\Sigma_S(\mathcal{C})$ means its degree multiplied by the degree of the rational map $\Phi_{F,S}$ restricted to $\mathcal{C}$. The class with multiplicity of $\Sigma_{\mathcal{S}}(\mathcal{C})$ means its class multiplied by the degree of the rational map $\rho_{F,S}$ restricted to $\mathcal{C}$. Our formulas complete the formula obtained by Chasles in [3] for the class of a caustic by reflection (for a generic $\mathcal{C}$ and a generic S). Let us indicate that, in [1], Brocard and Lemoyne gave, without any proof, formulas for the degree and class of caustics by reflection (for a Plücker curve $\mathcal{C}$ and for S not at infinity). It seems that their formulas come from an incorrect composition of formulas

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by Salmon and Cayley [10] for some caracteristic invariants of pedal and evolute curves (using the representation of caustics by reflection given by Quetelet and Dandelin). This is discussed in [8]. Let us also mention the work of Catanese and Trifogli on focal loci, which generalize evolutes to higher dimension [11, 2].

The question of the birationality of the rational maps $\rho_{F,S}$ and $\Phi_{F,S}$ on $\mathcal{C}$ is not evident even if S is not at infinity. Indeed, according to results of Quetelet and Dandelin [9, 4], when S is not at infinity, the caustic $\Sigma_S(\mathcal{C})$ is the evolute of the S-centered homothety (with ratio 2) of the pedal of $\mathcal{C}$ from S (i.e. the evolute of the orthotomic of $\mathcal{C}$ with respect to S). But we just know that the evolute map is birational for a generic algebraic curve (see [5] by Fantechi).

In this note, we prove the birationality on $\mathcal{C}$ of the maps $\rho_{F,S}$ and $\Phi_{F,S}$ for any irreducible algebraic curve $\mathcal{C} \subset \mathbb{P}^2$ of degree $d \geq 2$ and for a generic S in $\mathbb{P}^2$. This result enables us to establish simple formulas for the degree and class of caustics by reflection valid for any irreducible algebraic curves $\mathcal{C} \subset \mathbb{P}^2$ of degree $d \geq 2$ and for a generic S in $\mathbb{P}^2$. In this study, the cyclic points $I = [1 : i : 0]$ and $J = [1 : -i : 0]$ play a particular role. We will also use the canonical projection $\pi : \mathbb{C}^3 \setminus \{0\} \to \mathbb{P}^2$.

2. Birationality

Theorem 2.1. *Let $\mathcal{C} = V(F) \subset \mathbb{P}^2$ be any irreducible algebraic curve of degree $d \geq 2$. For a generic $S \in \mathbb{P}^2$, the maps $\rho_{F,S}$ and $\Phi_{F,S}$ are birational on $\mathcal{C}$.*

Before going into the proof of our Theorem, let us introduce some notations and recall some facts (see [7]). For any line $\mathcal{D} = V(ax + by + cz) \in \mathbb{P}^2$ such that $a^2 + b^2 \neq 0$, we define the orthogonal symmetric with respect to $\mathcal{D}$ as the rational map $\sigma_{\mathcal{D}} : \mathbb{P}^2 \to \mathbb{P}^2$ (which is an involution) given by

$$\sigma_{\mathcal{D}}[x : y : z] = \pi\left((a^2 + b^2) \cdot (x, y, z) + (ax + by + cz) \cdot (a, b, 0)\right).$$

Let $\mathcal{C} = V(F) \subset \mathbb{P}^2$ be an irreducible algebraic curve of degree $d \geq 2$ and let $S \in \mathbb{P}^2 \setminus \{I, J\}$ be a (light) position. We define $C_0 := \mathcal{C} \setminus V(F_x^2 + F_y^2)$. Observe that this set corresponds to the complement in $\mathcal{C}$ of the cyclic apparent contour of $\mathcal{C}$ (the cyclic apparent contour of $\mathcal{C}$ from the cyclic points). We recall that the reflected line $\mathcal{R}_{m,S,\mathcal{C}}$ at $m \in \mathcal{C}_0 \setminus \{S\}$ is the line $(m\, \sigma_{\mathcal{T}_m\mathcal{C}}(S))$, where $\mathcal{T}_m\mathcal{C}$ is the tangent to $\mathcal{C}$ at m. For any $m \in \mathcal{C}_0$, we define the normal line $\mathcal{N}_m\mathcal{C}$ to $\mathcal{C}$ at m as the line containing m and $[F_x(m) : F_y(m) : 0]$.

For any $m \in \mathcal{C}_0$, we consider the set K_m of points $S \in \mathbb{P}^2$ such that there exists $m' \in \mathcal{C}_0 \setminus \{m\}$ satisfying $\rho_{F,S}(m') = \rho_{F,S}(m) \neq 0$. Observe that the set $\mathcal{A}$ of $S \in \mathbb{P}^2$ such that $\rho_{F,S}$ is not birational can be written $\mathcal{A} = \bigcup_{E \subset \mathcal{C}_0 \,:\, \#E < \infty} \bigcap_{m \in \mathcal{C}_0 \setminus E} K_m$. To prove that $\rho_{F,S}$ is birational for a generic S in $\mathbb{P}^2$, we prove that $\mathcal{A}$ is contained in a subvariety of codimension at least 1 in $\mathbb{P}^2$. Our proof is based on the following lemma.

Lemma 2.2. *For any $m \in \mathcal{C}_0$, the set K_m is contained in a (possibly non irreducible) algebraic curve $\bar{K}_m$ of degree at most $2d^2 + 2$.*

Proof. Let us consider any $m \in \mathcal{C}_0$. Let $S \in K_m$ and $m' \in \mathcal{C}_0 \setminus \{m\}$ satisfying $\rho_{F,S}(m') = \rho_{F,S}(m) \neq 0$. Then $(m\, m') = \mathcal{R}_{m,S,\mathcal{C}} = \mathcal{R}_{m',S,\mathcal{C}}$ and so S is in

$\mathcal{A}_{m,m'} := \sigma_{\mathcal{T}_m\mathcal{C}}((m\,m')) \cap \sigma_{\mathcal{T}_{m'}\mathcal{C}}((m\,m'))$. Observe that, if $\sigma_{\mathcal{T}_m\mathcal{C}}((m\,m')) = \sigma_{\mathcal{T}_{m'}\mathcal{C}}((m\,m'))$, then these lines are $(m\,m')$ and so $(m\,m')$ is stable by $\sigma_{\mathcal{T}_m\mathcal{C}}$ and by $\sigma_{\mathcal{T}_{m'}\mathcal{C}}$. But $\mathcal{T}_m\mathcal{C}$ and $\mathcal{N}_m\mathcal{C}$ are the only lines containing m which are stable by $\sigma_{\mathcal{T}_m\mathcal{C}}$. Therefore, $\sigma_{\mathcal{T}_m\mathcal{C}}((m\,m')) = \sigma_{\mathcal{T}_{m'}\mathcal{C}}((m\,m'))$, implies that $(m\,m') \in \{\mathcal{T}_m\mathcal{C}, \mathcal{N}_m\mathcal{C}\} \cap \{\mathcal{T}_{m'}\mathcal{C}, \mathcal{N}_{m'}\mathcal{C}\}$. If $\{\mathcal{T}_m\mathcal{C}, \mathcal{N}_m\mathcal{C}\} \cap \{\mathcal{T}_{m'}\mathcal{C}, \mathcal{N}_{m'}\mathcal{C}\} = \emptyset$, then S is the only point of $\mathcal{A}_{m,m'}$, so S is equal to

$$\tau_m(m') := \pi\left((m \wedge \sigma_{\mathcal{T}_m\mathcal{C}}(m')) \wedge (m' \wedge \sigma_{\mathcal{T}_{m'}\mathcal{C}}(m))\right).$$

Notice that τ_m is a rational map with coordinates of degree $2d$. We obtain that S belongs to the Zariski closure of $\tau_m(\mathcal{C})$, which (according to [6, Proposition 4.4]) is contained in an algebraic curve of degree at most $\mathcal{C} \cdot \tau_m^*(H) \leq 2d^2$ (where H is the hyperplane class in $\mathbb{P}^2$). Otherwise, $S \in \mathcal{A}_{m,m'} = (m\,m') \in \{\mathcal{T}_m\mathcal{C}, \mathcal{N}_m\mathcal{C}\}$. Finally, we have $K_m \subseteq \bar{K}_m := \overline{\tau_m(\mathcal{C})} \cup \mathcal{T}_m\mathcal{C} \cup \mathcal{N}_m\mathcal{C}$ which is an algebraic curve of degree at most $2d^2 + 2$ (use for example the fundamental lemma of [7] applied with τ_m). $\square$

Proof of Theorem 2.1. Let us prove that $\rho_{F,S}$ is birational on $\mathcal{C}$ for a generic S in $\mathbb{P}^2$. The birationality of $\Phi_{F,S}$ will follow. Indeed, for a generic S in $\mathbb{P}^2$, the caustic $\Sigma_S(\mathcal{C})$ is a curve (see for example [8]). Therefore, for generic $m, m' \in \mathcal{C}$, $\Phi_{F,S}(m) = \Phi_{F,S}(m')$ implies that $\rho_{F,S}(m) = \rho_{F,S}(m')$. With the notations of Lemma 2.2, we define $\mathcal{A}' := \bigcup_{E \subset \mathcal{C}_0\,:\,\#E<\infty} \bigcap_{m \in \mathcal{C}_0 \setminus E} \bar{K}_m$. We prove that the set $\mathcal{F} := \left\{\bigcap_{m \in \mathcal{C}_0 \setminus E} \bar{K}_m,\ E \subset \mathcal{C}_0,\ \#E < \infty\right\}$ is inductive for the inclusion. Let $\left(\mathcal{F}_j := \bigcap_{m \in \mathcal{C}_0 \setminus E_j} \bar{K}_m\right)_{j \geq 1}$ be an increasing sequence of sets belonging to $\mathcal{F}$. Let us show that the union Z of these sets is also in $\mathcal{F}$. First $Z \subseteq \bigcap_{m \in \mathcal{C}_0 \setminus \bigcup_{i\geq 1} E_i} \bar{K}_m \subseteq \bar{K}_{m_0}$ for some fixed $m_0 \in \mathcal{C}_0 \setminus \bigcup_{i \geq 1} E_i$. Now $\bar{K}_{m_0}$ is the union of a finite number of irreducible algebraic curves $C_1, \ldots, C_p$. Let $i \in \{1, \ldots, p\}$ and let d_i be the degree of C_i. If $C_i \subseteq Z$, then there exists $N_i \geq 1$ such that $C_i \subseteq \mathcal{F}_{N_i}$. Assume now that $C_i \not\subseteq Z$. Then $(C_i \cap \mathcal{F}_j)_{j \geq 1}$ is an increasing sequence of finite sets containing at most $d_i \times (2d^2 + 2)$ points. Therefore, there exists $N_i \geq 1$ such that $(C_i \cap Z) \subseteq \mathcal{F}_{N_i}$. We conclude that $Z = \mathcal{F}_{\max(N_1,\ldots,N_p)}$ and so Z is in $\mathcal{F}$. So $\mathcal{F}$ is inductive.

From the Zorn lemma, either $\mathcal{F}$ is empty or it admits a maximal element (for the inclusion). If it is empty, then $\mathcal{A} = \mathcal{A}' = \emptyset$. If it is not empty and if $\mathcal{F}_0 := \bigcap_{m \in \mathcal{C}_0 \setminus E_0} \bar{K}_m$ (with $E_0 \subset \mathcal{C}_0$ and $\#E_0 < \infty$) is a maximal element of $\mathcal{F}$, then $\mathcal{A}' = \mathcal{F}_0$. Indeed, $\mathcal{A}'$ contains $\mathcal{F}_0$ by definition of $\mathcal{A}'$. Conversely, let $S \in \mathcal{A}'$, there exists $E \subset \mathcal{C}_0$ such that $\#E < \infty$ and such that $S \in \bigcap_{m \in \mathcal{C}_0 \setminus E} \bar{K}_m$. Hence $S \in \bigcap_{m \in \mathcal{C}_0 \setminus (E \cup E_0)} \bar{K}_m$. Since we also have $\bigcap_{m \in \mathcal{C}_0 \setminus E_0} \bar{K}_m \subseteq \bigcap_{m \in \mathcal{C}_0 \setminus (E \cup E_0)} \bar{K}_m$, we conclude that $S \in \bigcap_{m \in \mathcal{C}_0 \setminus E_0} \bar{K}_m$. Therefore, in any case, $\mathcal{A}$ is contained in an algebraic curve, this gives the S-genericity of the birationality of $\rho_{F,S}$ and so the statement of Theorem 2.1. $\square$

3. Light generic formulas for the degree and the class of caustics

Let $\mathcal{C} = V(F) \subset \mathbb{P}^2$ be any irreducible algebraic curve of degree $d \geq 2$. We call **isotropic tangent** to $\mathcal{C}$ any tangent to $\mathcal{C}$ containing I or J. Before stating our formulas, let us introduce some notations.

For any $P \in \mathbb{P}^2$, we write $\mu_P(\mathcal{C})$ for the multiplicity of $\mathcal{C}$ at P. We recall that $\mu_P(\mathcal{C}) = 1$ means that P is a non singular point of $\mathcal{C}$. For any $P \in \mathcal{C}$, we write $\mathrm{Branch}_P(\mathcal{C})$ for the set of branches of $\mathcal{C}$ at P. Let us write $\mathcal{E}_{\mathcal{C}}$ for the set of couples $(P, \mathcal{B})$ with $P \in \mathcal{C}$ and with $\mathcal{B} \in \mathrm{Branch}_P(\mathcal{C})$. For any $(P, \mathcal{B}) \in \mathcal{E}_{\mathcal{C}}$, we write $\mathcal{T}_P\mathcal{B}$ for the tangent line to $\mathcal{B}$ at P and $e_{\mathcal{B}}$ for the multiplicity of $\mathcal{B}$. We recall that $\sum_{\mathcal{B}\in \mathrm{Branch}_P(\mathcal{C})} e_{\mathcal{B}} = \mu_P(\mathcal{C})$. For any $(P, \mathcal{B}) \in \mathcal{E}_{\mathcal{C}}$ and any algebraic curve $\mathcal{C}'$, we denote by $i_P(\mathcal{C}, \mathcal{C}')$ (resp. $i_P(\mathcal{B}, \mathcal{C}')$) the intersection number of $\mathcal{C}$ (resp. $\mathcal{B}$) with $\mathcal{C}'$ at P. We recall that the contact number $\Omega_{m_1}(\mathcal{C}, \mathcal{C}')$ of $\mathcal{C}$ with $\mathcal{C}'$ is given by $\Omega_{m_1}(\mathcal{C}, \mathcal{C}') = i_P(\mathcal{C}, \mathcal{C}') - \mu_P(\mathcal{C})\mu_P(\mathcal{C}')$. The line at infinity of $\mathbb{P}^2$ is written ℓ_∞. Combining Theorem 2.1 with the main results of [7, 8], we obtain:

Proposition 3.1. *Let $\mathcal{C} = V(F) \subseteq \mathbb{P}^2$ be any irreducible algebraic curve of degree $d \geq 2$ and of class $d^\vee$. For a generic $\mathcal{S} \in \mathbb{P}^2$, we have*

$$\deg(\Sigma_S(\mathcal{C})) = 3d + f_0 - t_I - t_J \quad \text{and} \quad class(\Sigma_{\mathcal{S}}(\mathcal{C})) = 2d^\vee + d - g - \mu_I(\mathcal{C}) - \mu_J(\mathcal{C}),$$

where g is the contact number of $\mathcal{C}$ with ℓ_∞, i.e. $g := \sum_{m_1\in\mathcal{C}\cap\ell_\infty} \Omega_{m_1}(\mathcal{C}, \ell_\infty)$, where f_0 is the number of "inflectional branches" of $\mathcal{C}$ not tangent to the line at infinity, i.e.

$$f_0 := \sum_{(P,\mathcal{B})\in\mathcal{E}_{\mathcal{C}}\,:\, i_P(\mathcal{B},\mathcal{T}_P\mathcal{B})>2e_{\mathcal{B}},\, \mathcal{T}_P\mathcal{B}\neq\ell_\infty} (i_P(\mathcal{B}, \mathcal{T}_P\mathcal{B}) - 2e_{\mathcal{B}}).$$

and where t_P is the number of branches of $\mathcal{C}$ tangent at P to the line at infinity: $t_P = \sum_{\mathcal{B}\in Branch_P(\mathcal{C})\,:\, \mathcal{T}_P\mathcal{B}=\ell_\infty} e_{\mathcal{B}}$.

Proof. According to Theorem 2.1, for a generic S in $\mathbb{P}^2$, the degree and class (with multiplicity) of $\Sigma_S(\mathcal{C})$ are equal to its degree and class. For the degree formula, we use Theorem 20 of [7]. For the class formula, we use Theorem 2 of [8]. For a generic $S \in \mathbb{P}^2$ ($S \in \mathbb{P}^2 \setminus (\mathcal{C} \cup \ell_\infty)$ not contained in an isotropic tangent to $\mathcal{C}$), we have $f = \mu_I(\mathcal{C}) + \mu_J(\mathcal{C})$, $f' = 0$, $g' = 0$ and $q' = 0$ (with the notations of Theorem 2 in [8]). □

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References

[1] H. Brocard, T. Lemoyne. Courbes géométriques remarquables. Courbes spéciales planes et gauches. Tome I. (French) Nouveau tirage Librairie Scientifique et Technique Albert Blanchard, Paris (1967) viii+451 pp.

[2] F. Catanese and C. Trifogli. *Focal loci of algebraic varieties. I. Special issue in honor of Robin Hartshorne*. Comm. Algebra 28 (2000), no. 12, pp. 6017–6057.

[3] M. Chasles. *Détermination, par le principe des correspondances, de la classe de la développée et de la caustique par réflexion d'une courbe géométrique d'ordre m et de classe n* Nouv. Ann. Math. 2 ser. vol. 10 (1871), p. 97–104, extrait C. R. séances A. S. t. LXII.

[4] G. P. Dandelin. *Notes sur les caustiques par réflexion*, (1822).

[5] B. Fantechi. *The Evolute of a Plane Algebraic Curve*, (1992) UTM 408, University of Trento.

[6] W. Fulton. *Intersection Theory.* 2nd edition (1998) Springer Verlag.

[7] A. Josse, F. Pène. *On the degree of caustics by reflection.* Accepted for publication in Communications in Algebra.

[8] A. Josse, F. Pène. *On the class of caustics by reflection*. Preprint. Arxiv 1210.6551.
[9] L. A. J. Quetelet, *Énoncés de quelques théorèmes nouveaux sur les caustiques*. C. G. Q. (1828) vol 1, p.14, p. 147-149.
[10] G. Salmon G, *A treatise on higher plane curves: Intended as a sequel to a treatise on conic sections*. Elibron classics (1934).
[11] C. Trifogli. *Focal Loci of Algebraic Hypersurfaces: a General Theory*, Geom. Dedicata 70 (1998), pp. 1–26.

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